# Elastic Charge Form Factors of $\pi$ and $K$ Mesons

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Abstract

The elastic charge form factors of the charged $\pi$ and $K$ mesons are calculated in modified impulse approximation using instant form of relativistic Hamiltonian dynamics. Our approach gives pion and kaon electromagnetic form factors in the large range of momentum transfer. The results are in good agreement with the available data. Relativistic effects are large at all values of momentum transfers.

The pion and kaon form factors at large $Q^2$ depend strongly on the choice of model. The experiments on pion form factor at large momentum transfer planned at CEBAF will choose between such models. In the case of kaon such a choosing may be performed only if supplemented by accurate measurements of kaon MSR.

In recent years the interest has been renewed [1]-[6] to the study of electromagnetic form factors of pseudoscalar mesons and, particularly, of $\pi$ and $K$ mesons. This fact is due, first of all, to the experiments planned at CEBAF and concerning the measurement of pion (E-93-021) and kaon (E-93-018) charge form factors in the range of momentum transfer $Q^2 < 3$ GeV$^2$. It is possible that these future experiments will enable to choose between different theoretical models whose predictions differ rather strongly.

Such a difference of theoretical results seems to be quite natural, because one encounters a lot of difficulties while calculating the structure of composite particles containing light quarks. The main difficulties are the following two. First, the importance of relativistic effects [2], [6] in the whole range of momentum transfer, including $Q^2 \approx 0$ (the value of mean square radius (MSR)) [5]. Second, the problem of calculation of the "soft" and "hard" structure. The hard part can be calculated from perturbative QCD. The soft part, which describes the structure at long and intermediate distances, is modeled in a variety of nonperturbative ways. The relativistic Hamiltonian dynamics (RHD) (for a review see [7]) is one of such ways. RHD unifies potential approach to composite systems and the condition of Poincaré-invariance. This method is based on the direct realization of the Poincaré group algebra or, in other words, of the relativistic invariance condition in the few body Hilbert space. RHD can be formulated in different ways ( different relativistic dynamics). At present the light front dynamics is the most popular one and widely used to the calculation of pion and kaon electromagnetic structure (see for example [2] - [4], [6]).

In this Letter we present a relativistic treatment of the problem of soft electromagnetic structure in the framework of alternative form of relativistic dynamics, the so called instant form (IF) of RHD. Our approach, which provides good description of the available data for the elastic charge form factors for the charged $\pi$ and $K$ mesons, will be briefly outlined here. The details are partially given in [8] and will be given elsewhere. The IF of relativistic dynamics, although not widely used, has some advantages: first, the calculations can be performed in a straightforward way; second, this approach is obviously rotationally invariant (this fact is particularly important for treating spin effects and polarization effects). However, the approach encounters some difficulties of general kind – the problem of electromagnetic current conservation and of its relativistic covariance [9] - [11].

We hope that we have found the way to overcome these difficulties.

Let us consider $\pi$ meson and $K$ meson as quark ($q$) — antiquark ($\bar{Q}$) composite system. We shall use different quark masses $M_q$ and $M_{\bar{Q}}$ as in $K$ meson. The results for $\pi$ meson can be obtained if $M_q = M_{\bar{Q}}$.

The charge form factor for two-quark system can be obtained from the electromagnetic current matrix element for composite system

$$< p_c |\, j_\mu \,| p'_c >= (p_c + p'_c)_\mu \, F_c(Q^2), \qquad (1)$$

$F_c(Q^2)$ – electromagnetic form factor of composite system, $p$ – 4-momentum of system.

We shall act following the basic assumptions, valid for all forms of dynamics in RHD [7]. The RHD is based on the including of the operator, describing $q\bar{Q}$ interaction in the generators of Poincaré group while the commutation relations of Poincaré algebra are fulfilled. One usually includes $\hat{U}$ in the mass square operator of the free two particle system in additive way [7]: $P^2 = (p_1+p_2)^2 \to \hat{M}_I^2 = P^2+\hat{U}$. In the case of IF dynamics the Poincaré algebra is conserved if $\hat{U}$ commutes with the total angular momentum operator $\hat{\vec{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$, with the operator of total 3-momentum $\hat{\vec{P}}$ and with the operator $\vec{\nabla}_P$. The complete set of commuting operators for the two-particle system with interaction contains now: $\hat{M}_I^2$, $\hat{J}^2$, $\hat{J}_3$, $\hat{\vec{P}}$. In the case of IF the operators $\hat{J}^2$, $\hat{J}_3$, $\hat{\vec{P}}$ coincide with the appropriate operators of the two-particle system without interaction and one can construct the basis in which these three operators are diagonals. While working in this basis to obtain the wave function one needs to diagonalize $\hat{M}_I^2$.

In RHD the Hilbert space of composite particle states is the tensor product of single particle Hilbert spaces: $\mathcal{H}_{q\bar{Q}} \equiv \mathcal{H}_q \otimes \mathcal{H}_{\bar{Q}}$ and the state vector in RHD is a superposition of two-particle states. As a basis in $\mathcal{H}_{q\bar{Q}}$ one can choose the following set of vectors:

$$\begin{aligned} |\,\vec{p}_1\,, m_1; \vec{p}_2\,, m_2 >= |\,\vec{p}_1\,, m_1 > \otimes |\,\vec{p}_1\,, m_2\ >, \\ <\vec{p}\,, m\,|\,\vec{p}\,'\,, m' >= 2p_0\,\delta(\vec{p}-\vec{p}\,')\,\delta_{mm'}\ , \end{aligned} \tag{2}$$

Here $\vec{p}_1$ , $\vec{p}_2$ — are particle momenta, $m_1$ , $m_2$ — spin projections.

Since we consider the two-quark system as one composite system, then the natural basis is one with separated center-of-mass motion:

$$|\,\vec{P},\ \sqrt{s},\ J,\ l,\ S,\ m_J\ >, \tag{3}$$

with $P_\mu = (p_1+p_2)_\mu$, $P_\mu^2 = s$, $\sqrt{s}$ — the invariant mass of two-particle system , $l$ — the angular momentum in the center-of-mass frame, $S$ — total spin, $J$ — total angular momentum, $m_J$ — projection of total angular momentum.

The basis (3) is connected with (2) through the Clebsch – Gordan decomposition of the Poincaré group. The details of this procedure one can find in [12]. Now the decomposition of the electromagnetic current matrix element for the composite system (1) in the basis (3) has the form

$$\begin{aligned} (p_c+p_c')_\mu\,F_c(Q^2) = \sum \int \frac{d\vec{P}}{N_{C-G}}\,\frac{d\vec{P}'}{N'_{C-G}}\,d\sqrt{s}\,d\sqrt{s'} < p_c|\vec{P},\sqrt{s},J,l,S,m_J > \cdot \\ <\vec{P},\sqrt{s},J,l,S,m_J\,|\,j_\mu\,|\,\vec{P}',\sqrt{s'},J',l',S',m_J{}' > \cdot \\ <\vec{P}',\sqrt{s'},J',l',S',m_J{}'|p_c{}' > . \end{aligned} \tag{4}$$

Here the sum is over the discrete variables of the basis (3).
$<\vec{P}\,,\ \sqrt{s},\ J,\ l,\ S,\ m_J|p_c>$ - is the composite system wave function

$$<\vec{P}\,',\ \sqrt{s'},\ J',\ l',\ S',\ m_J'|\,p_c >= N_c\delta(\vec{P}\,'-\vec{p}_c)\delta_{JJ'}\delta_{m_Jm_J'}\delta_{ll'}\delta_{SS'}\,\varphi^J_{lS}(k)\ . \tag{5}$$

$k = \sqrt{(s^2 - 2s(M_{\bar s}^2 + M_u^2) + \eta^2)/4s}\,, \quad \eta = M_q^2 - M_{\bar Q}^2$. $N_c, N_{C-G}$ are factors due to normalization. Concrete form of $N_c$ and $N_{C-G}$ will not be used.

Let us discuss the current operator matrix element which enters the r.h.s. of the equation (4).

In the case of non-interacting quark system electromagnetic current matrix element of this system can be parametrized similarly to the standard case of one-particle matrix element, e.g. for meson (1), i.e. it is possible to extract the invariant part – form factor $g_0$:

$$< \vec{P}, \sqrt{s}, J, l, S, m_J |\, j^0_\mu \,| \vec{P}', \sqrt{s'}, J', l', S', m_J{}' >= \\ = A_\mu(s, Q^2, s')\, g_0(s, Q^2, s'). \quad (6)$$

The vector $A_\mu(s, Q^2, s')$ is defined by the current transformation properties (by the Lorentz–covariance and the current conservation law):

$$A_\mu = (1/Q^2)[(s - s' + Q^2)P_\mu + (s' - s + Q^2)P'_\mu]. \quad (7)$$

In our parametrization the current is conserved by construction:

$$A_\mu(s, Q^2, s')Q^\mu = 0. \quad (8)$$

In the frame of basis (2) non-interacting current matrix element has the following form:

$$< \vec{p}_1, m_1; \vec{p}_2, m_2 | j^0_\mu | \vec{p}'_1, m'_1; \vec{p}'_2, m'_2 >= \\ =< \vec{p}_1, m_1 | \vec{p}'_1, m'_1 >< \vec{p}_2, m_2 | j_\mu | \vec{p}'_2, m'_2 > +(1 \leftrightarrow 2). \quad (9)$$

This is, as a matter of fact, the relativistic impulse approximation. The one-particle current in (9) is expressed in terms of one-quark form factors. Clebsh-Gordan decomposition of the basis (3) into basis (2) gives the expression of free form factor $g_0(s, Q^2, s')$ in terms of one-quark form factors:

$$g_0(s, Q^2, s') = \frac{\sqrt{ss'}}{\sqrt{[s^2 - 2s(M_{\bar s}^2 + M_u^2) + \eta^2][s'^2 - 2s'(M_{\bar s}^2 + M_u^2) + \eta^2]}} \cdot \\ \cdot \frac{Q^2(s + s' + Q^2)}{2[\lambda(s, -Q^2, s')]^{3/2}} \cdot \Big(B^u(s, Q^2, s') + B^{\bar s}(s, Q^2, s')\Big)\,, \quad (10)$$

$$B^{\bar s}(s, Q^2, s') = \Big[f_1^{(\bar s)}(s + s' + Q^2 - 2\eta)\cos(\omega_1 + \omega_2) - \\ - f_2^{(\bar s)}\frac{M_{\bar s}}{2}\xi(s, Q^2, s')\sin(\omega_1 + \omega_2)\Big]\;\theta(s, Q^2, s')\,,$$

$$\xi(s, Q^2, s') = \sqrt{-\lambda(s, -Q^2, s')M_{\bar s}^2 + ss'Q^2 - \eta Q^2(s + s' + Q^2) + Q^2\eta^2}\,,$$

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)\,,$$

$$f_1^{(\bar s)} = \frac{2M_{\bar s}\, G_E^{(\bar s)}(Q^2)}{\sqrt{4M_{\bar s}^2 + Q^2}}; \qquad f_2^{(\bar s)} = -\frac{4\, G_M^{(\bar s)}(Q^2)}{M_{\bar s}\sqrt{4M_{\bar s}^2 + Q^2}}\,,$$

$$\omega_1 = \mathrm{arctg}\frac{\xi(\,s\,,Q^2\,,s')}{M_u[(\sqrt{s}+\sqrt{s'})^2+Q^2]+(\sqrt{s}+\sqrt{s'})(\sqrt{ss'}+\eta)}\;,$$

$$\omega_2 = \mathrm{arctg}[(\sqrt{s}+\sqrt{s'}+2M_{\bar{s}})\,\xi(\,s\,,Q^2\,,s')\cdot$$

$$\{M_{\bar{s}}(s+s'+Q^2)(\sqrt{s}+\sqrt{s'}+2M_{\bar{s}})+\sqrt{ss'}(4M_{\bar{s}}^2+Q^2)-\eta[2M_{\bar{s}}(\sqrt{s}+\sqrt{s'})-Q^2]\}^{-1}]\;,$$

$$\theta(s,Q^2,s') = \vartheta(s'-s_1)-\vartheta(s'-s_2)\;,$$

Here $\vartheta$ is the standard step function, $G_E^{(\bar{s})}(Q^2)$ and $G_M^{(\bar{s})}(Q^2)$ – Sachs form factors, $\omega_1$ and $\omega_2$ – are the Wigner rotation parameters.

$$s_{1,2} = M_{\bar{s}}^2+M_u^2+\frac{1}{2M_{\bar{s}}^2}(2M_{\bar{s}}^2+Q^2)(s-M_{\bar{s}}^2-M_u^2)\mp$$

$$\mp\frac{1}{2M_{\bar{s}}^2}\sqrt{Q^2(4M_{\bar{s}}^2+Q^2)[s^2-2s(M_{\bar{s}}^2+M_u^2)+\eta^2]}\;.$$

Function $B^u(s,Q^2,s')$ can be deduced from $B^{\bar{s}}(s,Q^2,s')$ by substitution $M_{\bar{s}}\leftrightarrow M_u$.

Let us return now to the Eq.(4). The current matrix element entering the r.h.s. of Eq.(4) must be interaction dependent. This dependence is known [2], [9] to be a consequence of the current conservation law and of the condition of current relativistic covariance. This means that we can not use in Eq. (4) the parametrization of non-interacting current matrix element (6) directly and need to include the interaction. Let us perform the interaction including in (6) in minimal manner: we shall include the interaction only in the vector function $A_\mu(s,Q^2,s')$ in Eq.(7):

$$A_\mu(s,Q^2,s')\;\rightarrow\;\frac{N_{C-G}N'_{C-G}}{N_cN'_c}A_\mu^{int}$$

$$A_\mu^{int} = A_\mu(s,Q^2,s')\Big|_{P_\mu\to p_{c\mu},\;P'_\mu\to p'_{c\mu}} = (p'_c+p_c)_\mu\;\;,$$

$$g_0(s,Q^2,s')\;\rightarrow\;g(s,Q^2,s') = g_0(s,Q^2,s')\;. \tag{11}$$

The function $A_\mu^{int}$ contains the interaction through the impulses $p'_{c\mu}$ and $p_{c\mu}$. Using (4), (6) and (11) we obtain now the following expression for the form factor:

$$F_c(Q^2) = \int d\sqrt{s}\;d\sqrt{s'}\;\varphi(k)\,g_0(s,Q^2,s')\,\varphi(k'). \tag{12}$$

Here we use for simplicity the notation: $\varphi_{lS}^J(k)\;\rightarrow\;\varphi(k)$.

Let us emphasize , that the r.h.s. of Eq.(4) with (11) inserted satisfies the current conservation law: it is orthogonal to the vector $Q_\mu=(p'_c-p_c)_\mu$. This latter fact is rather noticeable because generally the construction of the conserved current operator for composite systems presents a rather complicated problem which is not solved yet [11], [13]. Thus, the Eq.(12) takes into account the relativistic covariance and the current conservation law. This is right for any choice of the function $g(s,\,Q^2,\,s')$, including the expressions (10), (11) which we use here.

For $\varphi(k)$ one can use any phenomenological wave function, normalized using the relativistic density of states: $\varphi(k)=\sqrt{\sqrt{s}(1-\eta^2/s^2)}\,u(k)\,k,\;u(k)$ - is nonrelativistic phenomenological wave function.

Let us discuss the numerical results.

The Eq. (12) was used to calculate pion and kaon form factors. The following wave functions were utilized:

1. A gaussian or harmonic oscillator (HO) wave function (see e.g. [2])

$$u(k) = N_{HO} \exp\left(-k^2/2b^2\right) . \tag{13}$$

2. A power-law (PL) wave function (see e.g. [6], [3])

$$u(k) = N_{PL}\,(k^2/b^2+1)^{-n} \,, \quad n = 2\,,3\,. \tag{14}$$

3. The wave function with linear confinement from Ref.[14]:

$$u(r) = N_T\, \exp(-\alpha r^{3/2} - \beta r)\,, \quad \alpha = \frac{2}{3}\sqrt{2\, M_r\, a}\,, \quad \beta = M_r\, b\,. \tag{15}$$

$a\,,b$ – parameters of linear and Coulomb parts of potential respectively, $M_r$ – reduced mass of two-particle systems.

Our point of view is that the choice of parameters is to be done in such a way as to give the experimental value of MSR, because MSR can be determined model independently by direct experiment on elastic meson-electron scattering. Using the definition of MSR we obtain the condition:

$$< r^2 >= -\,6\, dF_\pi(Q^2)/dQ^2\Big|_{Q^2=0} =< r^2 >_{exp} \tag{16}$$

so that the parameters in question are not more independent [5]. They are connected through Eq. (16) which must be fulfilled within the experimental errors of r.h.s.

In our calculations we have fixed the masses of u- and d-quarks to be equal to 0.25 GeV, and that of s-quark from the approximate relation $M_s/M_u \approx 1.4$. These values are usually used in relativistic calculations. Once the masses are fixed the Eq. (16) fixes parameters $b$ in the models (13) and (14) or parameter $a$ in the model (15). We have used $b = 0.7867$ for the model (15) which means that the strong coupling constant is equal to 0.59 on the light meson mass scale. We have supposed the quark anomalous magnetic moment to be zero.

The results for pion form factor at large $Q^2$ are given on fig.1. If MSR is accurately given, then one can see that the results depend strongly on the choice of model. This dependence is much more pronounced than the dependence on the parameters variation in the frame of one chosen model within the experimental error of MSR in (16). This means that the experiments on pion form factor at large $Q^2$ planned to be realized on CEBAF in fact will enable one to choose between different two-quark pion models.

The situation with kaon form factor is quite different. This is due to the fact that kaon MSR is measured with rather less accuracy than that of pion ($\sim$ 15% and $\sim$ 2% respectively ) and then the Eq.(16) in the kaon case does not give such strong constraints on the parameters as in the case of pion. So the variations of the values of kaon form factor at large $Q^2$ compatible with (16) are large. One can see from the fig. 2 that the model dependence of kaon form factor is strong at large $Q^2$, however for the model discrimination one needs MSR and kaon form factor at low-momentum transfers to be

measured with much greater accuracy ($\sim 2\%$ as in the case pion). This problem can not be ignored in other approaches too.

It is possible to make the following conclusions:

1. The relativistic approach used in this Letter gives pion and kaon form factors in good agreement with the available data. Relativistic effects are large at all values of momentum transfers.

2. The pion and kaon form factors at large $Q^2$ depend strongly on the choice of model. The experiments on pion form factor at large momentum transfer planned at CEBAF will choose between such models.

3. In the case of kaon such a choosing may be performed only if supplemented by accurate measurements of kaon MSR.

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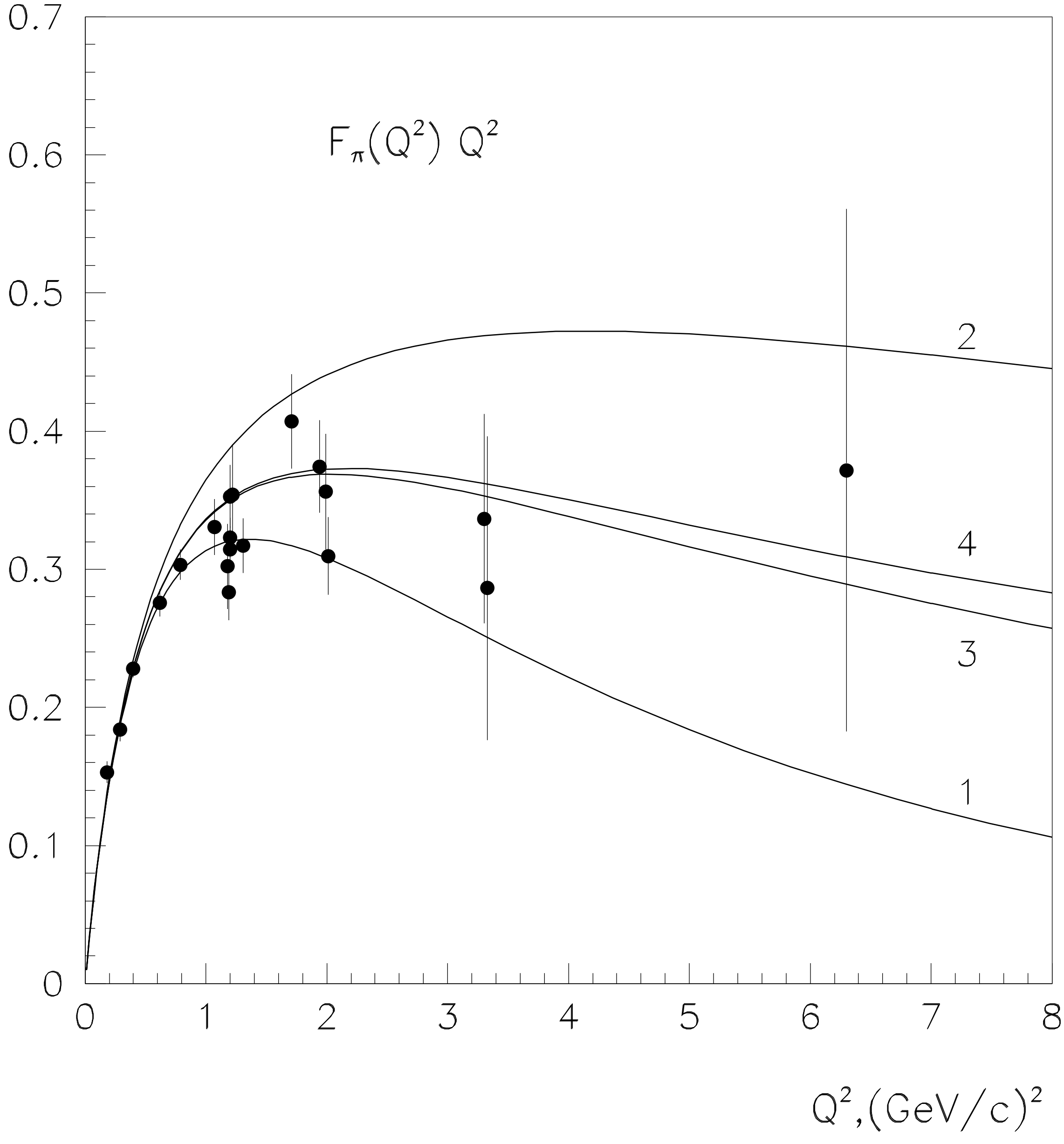
## Figure Captions

Fig. 1. Electromagnetic pion form factor, $Q^2 F_\pi(Q^2)$, at high momentum transfer in different models ($M_u = M_d = 0.25\ GeV/c$).
1 – harmonic oscillator wave function Eq.(13), $b = 0.207$ GeV;
2 – power-law wave function Eq.(14), $n = 2$, $b = 0.274$ GeV;
3 – power-law wave function Eq.(14), $n = 3$, $b = 0.388$ GeV;
4 – wave function Eq.(15) from model with linear confinement [14],
$a = 0.0183$ GeV$^2$, $b = 0.7867$.
Experimental data are taken from Ref. [15], [16].

Fig. 2. Electromagnetic kaon form factor at high momentum transfer in different models. The same line code as in Fig.1 is used. $M_s = 0.35$ GeV, 1 – $b = 0.255$ GeV, 2 – $b = 0.339$ GeV, 3 – $b = 0.480$ GeV, 4 – $a = 0.0318$ GeV$^2$, $b = 0.7867$.
Experimental data are taken from Ref. [17]



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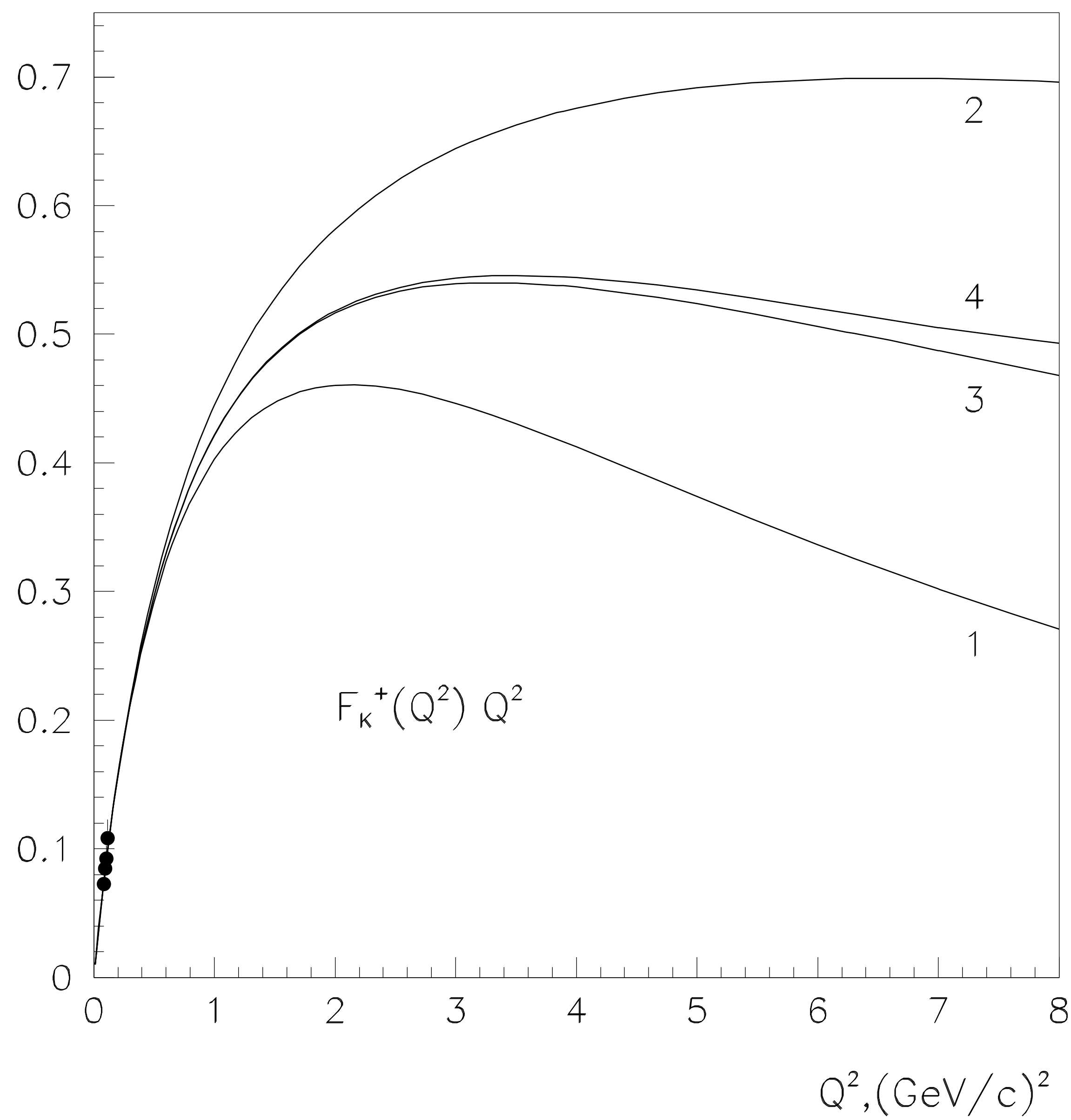
$Q^2,(GeV/c)^2$



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$F_{K^+}(Q^2)\,Q^2$
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